Maple Tutorial III

CS 310 — Fall 1999

This handout describes how to work with matrices within Maple, and how to use them to solve systems of equations. This tutorial is an extension of what is in Section 36 of your course notes.

1 Matrices in Maple

To enter a matrix into Maple, you use the `array` command. For example,

```
A := array( [ [1,2,3], [4,5,6] ] );
```

Maple responds by printing the matrix you entered,

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Notice that in Maple you use the "`[`" and "`]`" characters to enter matrices. The `array` command takes a list of lists, marked by `[...]`, and creates a matrix with each list forming a row of the matrix.

What do you think happens if you type: `A := array( [ [1, 2], [4, 5, 6] ] );`? Maple will respond with an error. Since the matrix isn't rectangular, Maple will not be able to create it.

To add or multiply matrices, you must use the `evalm` command. For example,

```
A := array( [ [1, 2], [3, 4] ] );
B := array( [ [-1, 3], [-4, 1] ] );
S := evalm( A + B );
P := evalm( A &* B );
```

You have to use "`&*`" to indicate multiplication. The matrix S is the matrix sum $A + B$, and the matrix P is the matrix product AB. Of course, in order to add or multiply two matrices the sizes must match up correctly. You should get

$$S = \begin{bmatrix} 0 & 5 \\ -1 & 5 \end{bmatrix}, \qquad P = \begin{bmatrix} -9 & 5 \\ -19 & 13 \end{bmatrix}$$

Here is another example, where the matrices contain variables in addition to numbers.

```
aa := array( [ [5, a], [2, 7] ] );
bb := array( [ [1, b], [2, a] ] );
ss := evalm( aa + bb );
pp1 := evalm( aa &* bb );
pp2 := evalm( bb &* aa );
```

Notice that since `pp1` and `pp2` are not the same, we can tell that matrix multiplication is not commutative (i.e., $aa * bb \neq bb * aa$).

To compute the square of a matrix, you can use `evalm( aa^2 );` This will produce the same answer as `evalm( aa &* aa );` Powers of matrices are only defined for square matrices — do you see why?

Here is an example where we compute some powers of a matrix.

```
A := array( [ [1, a], [b, 1] ] );
A2 := evalm( A^2 );
A4 := evalm( A2^2 );
iA := evalm( A^(-1) );
```

The value of `A2` is A^2, and `A4` is A^4. When you raise a matrix A to the power -1 you get a special matrix called the *inverse* of A. We'll discuss matrix inverses in the next section.

2 Matrix Inverses and the Identity Matrix

Try the following commands.

```
A := array( [ [1, 2, 5], [3, 1, 7], [6, 9, -1] ] );
A1 := evalm( A^(-1) );
P := evalm( A &* A1 );
```

If all goes well you should get

$$A1 = \begin{bmatrix} \frac{-64}{131} & \frac{47}{131} & \frac{9}{131} \\ \frac{45}{131} & \frac{-31}{131} & \frac{8}{131} \\ \frac{21}{131} & \frac{3}{131} & \frac{-5}{131} \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The product P is the 3×3 *identity* matrix. For any square matrix A, it is always true that $A * A^{-1}$ is the identity matrix. (It is also true that $A^{-1} * A$ is also the identity matrix.) Notice that even a simple matrix like A can have a really ugly inverse! Can you verify (by hand) that $A^{-1} * A$ is the identity?

Here is another example. This time we'll take the inverse of a 2×2 matrix, so we'll get the 2×2 identity matrix when we compute $A * A^{-1}$. (And $A^{-1} * A$.)

```
A := array( [ [a, b], [c, d] ] );
A1 := evalm( A^(-1) );
evalm( A &* A1 );
evalm( A1 &* A );
```

In general, the $N \times N$ identity matrix is:

$$I_N = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

The matrix elements along the diagonal are all 1, and everything else is all 0.

The identity matrix has the interesting property that whenever you multiply a matrix by it, you get the same matrix again. That is, $A * I = A$ and $I * A = A$.

```
Id := array( [ [1, 0], [0, 1] ] );
A := array( [ [-2, 9], [8, 13] ]  );
evalm( Id &* A );
evalm( A &* Id );
```

For the last two statements, Maple should return the matrix A again, since I was the 2×2 identity matrix.

3 Using Matrices to Solve Linear Systems

Many students are often confused by the "weird" definition of matrix multiplication. Why multiply the rows of the first matrix by the columns of the second matrix? Why not just multiply matrices in the same way that we add them — by simply multiplying each element together?

It turns out that with the weird definition, it makes it easy to express systems of linear equations with matrices.

Suppose we have a system of equations. For example,

$$\begin{aligned} 2x - y &= 5 \\ x - 3y &= -5 \end{aligned}$$

We want to solve these for x and y. We'll do it using matrices. We can re-write the system using matrices, as follows:

$$\begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

If you multiply the two matrices on the left-hand side together, and equate the top and bottom elements with the top and bottom elements of the right-hand side, you'll get the same system of equations we had above.

What good is this? Let's enter these matrices into Maple.

```
A := array( [ [2, -1], [1, -3] ] );
b := array( [ [5], [-5] ] );
X := array( [ [x], [y] ] );
```

The matrix equation for our system is $A * X = b$. Look what happens if we multiply this equation by the inverse of A:

$$\begin{aligned} A * X &= b \\ A^{-1} * A * X &= A^{-1} * b \\ I * X &= A^{-1} * b \\ X &= A^{-1} * b \end{aligned}$$

On the left-hand side, the $A^{-1} * A$ becomes the identity matrix I, and $I * X = X$ since multiplying a matrix by the identity is like multiplying a number by 1.

So, to solve the equations, we need to find the inverse of A, and multiply it by b. In Maple, we can do this as follows

(... previous example continued)

```
iA := evalm( A^(-1) );
evalm( iA &* b );
```

You should get the matrix $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Since this is equal to the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$, we have the solution $x = 4$ and $y = 3$. Notice that we didn't even really need the matrix `X` that we defined.

Let's do another example. We'll solve the 3 equations

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 3 \\ x & - & y & + & 2z & = & -3 \\ -2x & + & y & - & 3z & = & 4 \end{array}$$

We first form a matrix A where each row is the coefficients of the variables in one of the equations. Then, we form another matrix b which has only 1 column, and each row is the number from the right-hand side of the equations. We can then find the values of x, y and z by computing the 3 by 1 matrix $A^{-1}b$.

Notice that we don't need to enter the vector containing the variables, as long as we know which entry in the answer corresponds to which variable.

```
A := array( [ [1, 1, 1], [1, -1, 2], [-2, 1, -3] ] );
b := array( [ [3], [-3], [4] ] );
iA := evalm( A^(-1) );
evalm( iA &* b );
```

The solution is $x = -3, y = 4, z = 2$. Another way to solve the equations using Maple is with the `solve` command, like this:

```
eq1 := x + y + z = 3;
eq2 := x - y + 2*z = -3;
eq3 := -2*x + y - 3*z = 4;
sol := solve({eq1,eq2,eq3},{x,y,z});
```

☞ Using matrices, solve the system

$$\begin{array}{rcrcrcr} a & - & b & - & c & = & 1 \\ a & + & b & & & = & 0 \\ -2a & - & b & + & c & = & -2 \end{array}$$

(The correct answers are $a = -1, b = 1, c = -3$)

4 Problems

☞ Use Maple to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 6 & -3 \\ -4 & 15 & 0 \\ 7 & 3 & 2 \end{bmatrix}$$

Show that AA^{-1} and $A^{-1}A$ are each the identity matrix.

☞ Solve

$$\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for s_1 and s_2 as functions of a and b.

The answers are:

$$s_1 = \frac{a-1}{ab-1} \qquad s_2 = \frac{b-1}{ab-1}$$

☞ Using matrices, solve the equations:

$$\begin{aligned} ax - 2by &= -1 \\ bx + ay &= 1 \end{aligned}$$

for x and y as functions of a and b.

The answers are:

$$x = \frac{2b-a}{a^2+2b^2} \qquad y = \frac{a+b}{a^2+2b^2}$$

☞ Solve this system of linear equations for x, y and z as functions of a, b and c:

$$x + y + z = a$$

$$2x + 4y = b - a$$

$$c = z - 3y$$

The answers are:

$$\begin{aligned} x &= c - 2a + b \\ y &= \frac{3}{4}a - \frac{1}{4}b - \frac{1}{2}c \\ z &= \frac{9}{4}a - \frac{3}{4}b - \frac{1}{2}c \end{aligned}$$