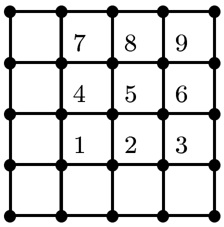
CS310 **Assignment 3** Fall 1999

Prof. Blanchard Due date: Oct. 26, 1999
Prof. Strikwerda Due date: Oct. 27, 1999

1. (30 points) Consider a square plate in which three sides are held at 0°C, and the fourth side is held at 100°C. The governing equation for the temperatures in the interior of the plate is the partial differential equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

This is called Laplace's equation and expresses the idea that the temperature at any point is the average of the temperatures in the neighborhood. We divide this plate up into a 4×4 grid as shown.



We take the side at 100°C to be the bottom side.

Using this grid and the approximation that the temperature at each point is the average of the temperature at the four neighbors in the grid we obtain the following linear system for the 9 unknown internal temperatures:

$$[A]\{x\} = \{b\},$$

where

$$[A] = \begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix}$$

$$\{x\} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} \quad \text{and} \quad \{b\} = \begin{pmatrix} -100 \\ -100 \\ -100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solve this system using Matlab and compare to the analytical solution ($T_5 = 25.00$, $T_1 = 43.2028$, and $T_2 = 54.0529$).

2. (10 points) Calculate the perimeter of an ellipse with a semi-major axis a of unit length and an ellipticity e of 0.5. (The ellipticity is the ratio of c to a, where c is the distance from the center of the ellipse to a focal point.) The perimeter is given by:

$$P = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \cos^2\theta} \; d\theta$$

Use the trapezoidal rule in Excel to get an error at most 10^{-4}. Compare to the result obtained using Maple.

3. (10 points) Determine the moment of inertia about the vertical axis of a quarter-ellipse. This quantity is given by:

$$I = \int_0^b \left(t - \frac{4b}{3\pi}\right)^2 a\sqrt{1 - \frac{t^2}{b^2}} dt$$

where a and b are the minor and major axes of the ellipse. Assume that $a = 2$ and $b = 4$.

4. (30 points) For this exerciseœ you are to work with a cryptographic system invented by Lester Hill in the 1920's. For this system, we encode the letters `A-Z` by the integers 0,1,2, ..., 25. That is, 0 stands for `A`, 1 for `B`, and so on. Two people wishing to correspond using this system agree on an $n \times n$ matrix M with integer entries and determinant 1. The message is written out in numerical form in blocks of n symbols. Each block is multiplied by M on the right and reduced modulo 26. These blocks are concatenated together to form the encrypted message which is then transmitted. The receiver can compute M^{-1} and invert the transformation to recover the message.

   Example: Let use choose $M = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$ and encrypt the message `UW`, which produces the block (20, 22). We compute

$$(\,20 \quad 22\,)\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} = (\,86 \quad 108\,) \rightarrow (\,8 \quad 4\,)$$

   so the message transmitted is `IE`. The receiver gets this message and computes

$$(\,8 \quad 4\,)\begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix} = (\,20 \quad -4\,) \rightarrow (\,20 \quad 22\,)$$

   which gives the original message.

a) Matrices of determinant 1 can be easily constructed by multiplying together triangular matrices with 1's on the diagonal. Use Maple to compute

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 7 & 1 & 0 & 0 & 0 \\ 25 & 6 & 1 & 0 & 0 \\ 5 & 9 & 21 & 1 & 0 \\ 11 & 1 & 17 & 23 & 1 \end{pmatrix} \begin{pmatrix} 1 & 11 & 3 & 17 & 23 \\ 0 & 1 & 13 & 3 & 20 \\ 0 & 0 & 1 & 24 & 7 \\ 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

b) Use Maple's inverse command to find M^{-1}. Reduce the entries of M and M^{-1} mod 26 to make the encryption and decryption matrices.

   Take the message `PLEASESENDMONEY` and encrypt it using the result of a). What string of letters results? Now decrypt it and verify that you get the original message back.

   Note: The number x reduced modulo N is written $x \bmod N$ and is the remainder after dividing by N. It is never negative; we always have $0 \le x \bmod N < N$.

   Note: Two equivalent ways to reduce a matrix `M` modulo 26 in Maple are with the commands `map( x -> x mod 26, M)` and `map( modp, M, 26 )`.

Check the course web page for instructions on handing in the assignment.